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# Scattering of two gravitating particles: classical approach

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**Abstract.** The scattering of two gravitating particles is studied, using a predictive system to first order in $G$. The centre of mass differential cross section is given. When one or both masses are taken to be zero, the scattering of light-scalar particles and light–light is obtained. All the results are in agreement with those based upon the quantised linearised theory.

## 1. Introduction

In a preceding paper (Portilla 1979), an approximate predictive system for two gravitating particles was obtained up to first order in $G$. The Hamilton–Jacobi momenta of the particles, the total four-momentum and the angular intrinsic momentum were deduced up to the same order.

We can expect these first-order results to be useful in the case of high velocities. In this paper a direct application of this result is made for the scattering of two gravitating particles, since in this problem the fast-motion condition may be fulfilled. We are limited to a large impact parameter, $d$, such that $Gm/d \ll 1$. Here $m$ is the mass of any one of the particles, and we use unities such that $c = 1$.

The study of two-particle scattering is greatly simplified taking into account that the four-momentum involved remains constant along the world lines of the particles (Lapiedra 1979). Therefore, we must simply establish the equality between the values of these quantities at past infinity (initial state) and the values at future infinity (final state).

Throughout this paper, the same notation and formalism developed by Portilla (1979) will be used. In particular we will label the Hamilton–Jacobi momenta by $p_a$ with $a = 1, 2$; the total four-momentum by $P = p_1 + p_2$; and the angular intrinsic momentum by $W$. In this reference the infinite past limits of these quantities have been given. They are

$$\lim_{x^2\to\infty_{\mathrm{P}}} p_a^\mu = \Pi_{a\mathrm{I}}^\mu \qquad \Pi_a^\mu = m_a(\mathrm{d}x_a^\mu/\mathrm{d}s_a) \qquad a = 1, 2 \tag{1.1}$$

$$\lim_{x^2\to\infty_{\mathrm{P}}} P^\mu = \Pi_{1\mathrm{I}}^\mu + \Pi^\mu{}_{2\mathrm{I}} \qquad \lim_{x^2\to\infty_{\mathrm{P}}} W^\mu = W_{0\mathrm{I}}^\mu$$

  

where I refers to the initial state, and $W_0$ is the angular intrinsic momentum corresponding to isolated particles,

$$W_0^\mu = M^{-1}\delta^{\mu\beta\gamma\delta}x_\beta \Pi_{1\gamma}\Pi_{2\delta}$$
$$M = (m_1^2 + m_2^2 + 2K)^{1/2}$$
$$x^\mu = x_1^\mu - x_2^\mu.$$

We also need the future limits, which are easily calculated, and the result is

$$\lim_{x^2\to\infty_F} P_1^\mu = \Pi_{1F}^\mu + G[2(K^2+\Lambda^2)/\Lambda h^2]h^\mu \tag{1.2}$$

$$\lim_{x^2\to\infty_F} P^\mu = \Pi_{1F}^\mu + \Pi_{2F}^\mu \tag{1.3}$$

$$\lim_{x^2\to\infty_F} W^\mu = W_{0F}^\mu \tag{1.4}$$

where F refers to the final state.

This paper is arranged as follows. In § 2 the centre of mass differential cross section is obtained even when the masses are comparable. In § 3 the case in which one or both masses have zero rest mass is considered, leading us to the description of some features of the gravitational interaction of light.

## 2. The centre of mass differential cross section

Let us first consider the total four-momentum and the angular intrinsic momentum. We have seen that they have, at future infinity, the same expressions as the ones corresponding to isolated particles. Then, from the conservation of the total four-momentum we get

$$\Pi_{1I}^\mu + \Pi_{2I}^\mu = \Pi_{1F}^\mu + \Pi_{2F}^\mu \tag{2.1}$$

and in the centre of mass frame, $\boldsymbol{P} = 0$, the usual relations are available.

$$\begin{aligned} &\Pi_{1I}^\mu + \Pi_{2I}^\mu = \Pi_{1F}^\mu + \Pi_{2F}^\mu \\ &|\boldsymbol{\Pi}_{1I}| = |\boldsymbol{\Pi}_{2I}| = |\boldsymbol{\Pi}_{1F}| + |\boldsymbol{\Pi}_{2F}| \\ &\Pi_{aI}^0 = \Pi_{aF}^0 = [m_a^2 + |\boldsymbol{\Pi}|^2]^{1/2} \\ &\boldsymbol{\Pi} = \boldsymbol{\Pi}_{1I}. \end{aligned} \tag{2.2}$$

In the centre of mass frame, and for simultaneous configurations $x_1^0 = x_2^0$, the initial and final intrinsic momentum three-vectors $\boldsymbol{W}_{0I}$ and $\boldsymbol{W}_{0F}$ can be written

$$\boldsymbol{W}_{0I} = \boldsymbol{X}_I \times \boldsymbol{\Pi}_{1I} \qquad \boldsymbol{W}_{0F} = \boldsymbol{X}_F \times \boldsymbol{\Pi}_{1F}. \tag{2.3}$$

The moduli of these three-vectors give us the initial and final impact parameters

$$|\boldsymbol{W}_{0I}| = |\boldsymbol{\Pi}|\, d_I \qquad |\boldsymbol{W}_{0F}| = |\boldsymbol{\Pi}|\, d_F. \tag{2.4}$$

Taking into account the conservation of the intrinsic momentum, it is clear that these parameters are equal, i.e. $d_I = d_F$. This is a consequence of the fact that to first order, the intrinsic momentum also contains, at future infinity, the expression corresponding to isolated particles.

Let us now consider the Hamilton–Jacobi momentum. As it is conserved along the world lines, we have

$$\lim_{x^2 \to \infty_P} P_1^\mu = \lim_{x^2 \to \infty_F} P_1^\mu \tag{2.5}$$

and using the expressions given in (1.1,2) we obtain

$$\Pi_{1I}^\mu = \Pi_{1F}^\mu + G[2(K^2 + \Lambda^2)/\Lambda h^2]h^\mu \tag{2.6}$$

where all the quantities on the right-hand side correspond to the final state. We must keep in mind that the scalar Lorentz $h = (h^2)$ is equal to the impact parameter $h_I = h_F = d$, in the centre of mass frame and for $x_1^0 = x_2^0$. Then the last equation (2.6) gives us the four-momentum transfer $\Delta^\mu = \Pi_{1F}^\mu - \Pi_{1I}^\mu$ as a function of the impact parameter:

$$\Delta^\mu = -G[2(K^2 + \Lambda^2)/\Lambda h^2]h^\mu. \tag{2.7}$$

As is well known (Sard 1970) the four-momentum transfer squared $\Delta^2$ is related to the scattering angle in the centre of mass frame as

$$\Delta^2 = 4|\mathbf{\Pi}|^2 \sin^2(\chi/2) \tag{2.8}$$

and to the kinetic energy $T_2$, acquired by the struck particle, initially at rest, as

$$\Delta^2 = 2m_2 T_2. \tag{2.9}$$

Squaring (2.7) and using (2.8) and (2.9) we obtain

$$h = G\frac{K^2 + \Lambda^2}{\Lambda|\mathbf{\Pi}|\sin(\chi/2)} \tag{2.10}$$

$$T_2 = \frac{2G^2(K^2 + \Lambda^2)^2}{m_2 \Lambda^2 h^2}. \tag{2.11}$$

From (2.10) we can evaluate the differential cross section in the centre of mass frame

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{G(K^2 + \Lambda^2)}{\Lambda|\mathbf{\Pi}|}\right)^2 \frac{1}{\sin^4(\chi/2)} \tag{2.12}$$

$$d\Omega = 2\Pi \sin\chi \, d\chi.$$

Let us express $K$, $\Lambda$, $|\mathbf{\Pi}|^2$ in terms of the energy in the centre of mass frame, $P^0$:

$$K = \tfrac{1}{2}[P^{02} - m_1^2 - m_2^2] \tag{2.13}$$

$$\Lambda^2 = \tfrac{1}{4}[P^{02} - (m_1 + m_2)^2][P^{02} - (m_1 - m_2)^2]$$

$$|\mathbf{\Pi}|^2 = \Lambda^2/P^{02}.$$

Finally, the cross section can be written in terms of $P^0$ as:

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{4}G^2 P^{02}\left(1 + \frac{(P^{02} - m_1^2 - m_2^2)^2}{[P^{02} - (m_1 + m_2)^2][P^{02} - (m_1 - m_2)^2]}\right)^2 \frac{1}{\sin^4(\chi/2)}. \tag{2.14}$$

This expression reduces, for low velocities, to the Rutherford nonrelativistic cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{Gm_1 m_2}{2mV_\infty^2}\right)^2 \frac{1}{\sin^4(\chi/2)}$$

where $m$ is the reduced mass $m_1m_2/(m_1+m_2)$, and $V_\infty = V_{1\mathrm{I}} - V_{2\mathrm{I}}$ is the initial relative velocity.

Let us observe the behaviour of formula (2.14) when the velocities of the particles are so great that we can consider $a = 1, 2$. The cross section reduces to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{G^2 P^{02}}{\sin^4(\chi/2)}. \tag{2.15}$$

We see that the cross section increases with the energy. This is completely different from the behaviour corresponding to the electromagnetic interaction. Bel (1976) and Lapiedra (1979) have obtained, in the framework of predictive electrodynamics, a cross section which in the limit considered above reduces to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e_1^2 e_2^2}{P^{02}\sin^4(\chi/2)}.$$

This strange dependence upon the energy may be explained intuitively as a consequence of the equivalence principle. This principle is based on the assumption that gravity becomes more important at higher energies, since on increasing the energy each particle will appear heavier to the others and the gravitational effects will be augmented. The scattering angle, for example, according to the above assumption must be an increasing function of the energy.

From formula (2.10) we can obtain the expression of this dependence

$$\sin\frac{\chi}{2} = \frac{GP^0}{h}\left(1 + \frac{(P^{02} - m_1^2 - m_2^2)^2}{[P^{02} - (m_1+m_2)^2][P^{02} - (m_1 - m_2)^2]}\right) \tag{2.16}$$

and for $P^0 \gg m_1;\ m_2$ we have

$$\sin\tfrac{1}{2}\chi \sim 2GP^0/h. \tag{2.17}$$

We shall finish this section with two comments about the quantum field calculations. Firstly, let us point out that, just as predictive electrodynamics gives the same relativistic differential cross section as the lower order in quantum electrodynamics (Lapiedra 1979), so the gravitational cross section obtained here is in agreement with the linearised quantum theory, see for example Gupta (1952) and Scadron (1979).

Secondly, there is a general result about the asymptotic behaviour of the differential cross section, assuming the dominance of one particle exchange of spin $J$ in the $t$-channel (Muirhead 1972). The result is

$$\mathrm{d}\sigma/\mathrm{d}t \propto S^{2J-2} \qquad \text{for } s \to \infty \text{ and } t \text{ fixed} \tag{2.18}$$

where $s$ and $t$ are the scalars defined by

$$s = -(\Pi_{1\mathrm{I}} + \Pi_{2\mathrm{I}})^2 = P^{02}$$
$$t = -(\Pi_{1\mathrm{F}} - \Pi_{1\mathrm{I}})^2 = -\Delta^2.$$

In the gravitational case, one graviton exchange corresponds to $J = 2$. Taking into account the relations

$$t = -s\sin^2\tfrac{1}{2}\chi$$
$$s = P^{02}$$

we can obtain from (2.15)

$$\mathrm{d}\sigma/\mathrm{d}t \propto S^2 \qquad s \to \infty,\ t \text{ fixed}$$

which coincides with (2.18) for $J = 2$.

## 3. Some features of gravitational light scattering

Let us consider now formulae (2.14) and (2.16) for fixed initial energies. We shall obtain a description of the light–light and light–scalar particle by making the limits $m_1 \to 0$, $m_2 \to 0$ and $m_1 \to 0$, $m_2 = \text{constant}$. So we have

### *3.1. Gravitational light–light scattering*

Differential cross section (in the centre of mass frame)

$$\mathrm{d}\sigma/\mathrm{d}\Omega = 64G^2K^2/\chi^4. \tag{3.1}$$

Scattering angle

$$\chi = 8GK/h, \tag{3.2}$$

$K$ being the photon energy. In both formulae we have taken

$$\sin\tfrac{1}{2}\chi \sim \tfrac{1}{2}\chi.$$

### *3.2. Gravitational light–scalar particle*

$$\mathrm{d}\sigma/\mathrm{d}\Omega = 16G^2P^{02}/\chi^4 \tag{3.3}$$

$$\chi = 4GP^0/h \tag{3.4}$$

$$P^0 = K + (K^2 + m_2^2)^{1/2}.$$

These results coincide nicely with those obtained by Westervelt (1970) starting from the gravitational field due to a delta-function photon (Westervelt 1965). As was pointed out by the same author (1970) the classical calculations agree, for small angles, with the ones obtained in the framework of the quantised linearised theory (Boccaletti 1969, Barker *et al* 1966, 1967).

If we interpret $m_2$, in the formula (3.4), as the mass of an astronomical object we have $\chi = 4m_2G/h$. This is just the value predicted by General Relativity using the Schwartschild metric, and assuming the photon to be moving along a null geodesic. From formula (2.11) we can obtain an interesting result. Once the particle and the photon have been scattered, the former, assumed initially at rest, acquires a kinetic energy given by (2.11). One can obtain the values of $K^2$ and $\Lambda^2$, by substituting into (2.13) $m_1 = 0$ and $(P^0)^2 = m_2^2 + 2m_2E_1$, where $E_{\mathrm{I}}$ is the initial photon energy in the rest frame of the particle. So we have

$$K^2 = \Lambda^2 = m_2^2E_{\mathrm{I}}^2. \tag{3.5}$$

From (2.11) and (3.5) we get

$$T_2 = 8G^2m_2E_{\mathrm{I}}^2/h^2. \tag{3.6}$$

However, $T_2$ is just the diminution of the photon energy, therefore (3.6) gives us a redshift of second order in $G$

$$\frac{\nu_{\rm I}-\nu_{\rm F}}{\nu_{\rm I}}=\frac{8G^2m_2^2}{h^2}\frac{E_{\rm I}}{m_2} \tag{3.7}$$

$$E_{\rm I(F)} = (\text{Planck's constant}) \times \nu_{\rm I(F)}.$$

Obviously, if we take for $m_2$ the mass of an astronomical object then (3.7) will be a very small quantity. However, if the struck particle has an initial velocity $\pm u$ perpendicular to and directed towards the light path, one gets a redshift of first order in $G$. Let us sketch the calculation.

Taking into account the conservation of $p_2^\mu$, we can obtain the components of $\Pi_{2\rm F}^\mu$ in the initial rest frame of particle two. They are

$$\Pi_{2\rm F}^0 = m_2 + T_2 = m_2 + \mathrm{O}(G^2)$$

$$\boldsymbol{\Pi}_{2\rm F} = \frac{4Gm_2E_{\rm I}}{|\boldsymbol{h}|^2}\boldsymbol{h} = \frac{(2m_2T_2)^{1/2}}{|\boldsymbol{h}|}\boldsymbol{h} \tag{3.8}$$

where the three-vector $\boldsymbol{h}$ is orthogonal to $\boldsymbol{\Pi}_{1\rm I}$.

The zero-component in a frame $R_2'$, moving uniformly with velocity $\mp u$ directed towards the light path, is:

$$\Pi_{2\rm F}^{0'} = (1-u^2)^{-1/2}(\Pi_{2\rm F}^0 \pm u|\boldsymbol{\Pi}_{2\rm F}|). \tag{3.9}$$

Neglecting terms of second order in $G$ we get

$$\Pi_{2\rm F}^{0'} = (1-u^2)^{-1/2}[m_2 \pm u(2m_2T_2)^{1/2}] + \mathrm{O}(G^2) \tag{3.10}$$

and keeping in mind that

$$\Pi_{2\rm I}^{0'} = m_2(1-u^2)^{-1/2}$$

we obtain the energy acquired by the particle

$$\Pi_{2\rm F}^{0'} - \Pi_{2\rm I}^{0'} = \pm u(1-u^2)^{-1/2}(2m_2T_2)^{1/2} \tag{3.11}$$

which must coincide with the energy lost by the photon

$$E_{\rm I}' - E_{\rm F}' = \frac{\pm u}{(1-u^2)^{1/2}}\frac{4m_2GE_{\rm I}'}{h}. \tag{3.12}$$

We have taken into account that in this case $E_{\rm I}' = E_{\rm I}$. A first-order shift follows from (3.12):

$$\frac{\nu_{\rm I}'-\nu_{\rm F}'}{\nu_{\rm I}'} = \frac{\pm u}{(1-u^2)^{1/2}}\frac{4m_2G}{h}. \tag{3.13}$$

Westervelt (1971, 1969) reported a shift of $\pm u4m_2G/h$ which is in agreement with (3.13) for $u \ll 1$. He describes the interaction of light with matter by means of the Landau pseudotensor density.

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